IMPROVED BOUNDARY ELEMENTS IN TORSION PROBLEMS

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INTRODUCTION

Since the epoch-making "memoir" of Saint-Venant (1) in 1855 the torsion of prismatic and cilindrical bars has reduced to a mathematical problem: the calculation of an analytical function satisfying prescribed boundary values.

For over one century, till the first applications of the F.E.M. to the problem (ref. 2,3), the only possibility of study in irregularly shaped domains was the beatiful, but limitated, theory of complex function analysis (ref. 4, 5, 6), several functional approaches (ref.7) and the finite difference method (ref.7,8).

Nevertheless in 1963 Jaswon (9) published an interesting paper which was nearly lost between the splendid F.E.M. boom.The method was extended by Rizzo (ref.10) to more complicated problems and definitively incorporated to the scientific community background through several lecture-notes of Cruse (ref.11) recently published, but widely circulated during past years.

The work of several researches (ref.12, 13, 14, 15) has shown the tremendous possibilities of the method which is today a recognized alternative to the well established F.E. procedure. In fact, the firt comprehensive attempt to cover the method, has been recently published in textbook form (ref.16).

This paper is a contribution to the implementation of a difficulty which arises if the isoparametric elements concept is applicated to plane potential problems with sharp corners in the boundary domain.

In previous works, these problems was avoided using two principal approximations: equating the fluxes round the corner or establishing a binode element (in fact, truncating the corner).(ref.17).

The first approximation distortes heavily the solution in the

corner neighbourhood, and a great amount of element is neccesary to reduce its influence. The second is better suited (ref.18) but the price payed is increasing the size of the system of equations to be solved.

In this paper an alternative formulation, consistent with the shape function chosen in the isoparametric representation, is presented. For ease of comprehension the formulation has been limited to the linear element. Nevertheless its extension to more refined elements is straight forward.

Also a direct procedure for the assembling of the equations is presented in an attempt to reduce the in-core computer requirements

SAINT-VENANT TORSION

Experience shows that the torsion of prismatic bodies produces a warping of its transversal sections.

Assuming the solution is that obtained in circular cylinders plus a warping function.

$$\theta.\psi(x,y)$$

where θ is the unitarian angle of twist, SAINT-VENANT used as a trial, the following field of displacements

$$\begin{aligned} u &= -\theta yz \\ v &= \theta xy \\ w &= \theta.\psi(x,y) \end{aligned} \qquad (1)$$

where the z-axis is directed along the generators of the prism.

From (u, v, w) one can obtain the strains

$$\begin{aligned} \varepsilon_x &= \varepsilon_y = \varepsilon_z = 0 = \gamma_{xy} \\ \gamma_{xz} &= \theta(-y + \frac{\partial \psi}{\partial x}) \\ \gamma_{yz} &= (x + \frac{\partial \psi}{\partial y}) \end{aligned} \qquad (2)$$

and, through the LAME relations

$$\begin{aligned} \sigma_x &= \sigma_y = \sigma_z = 0 = \tau_{xy} \\ \tau_{xz} &= G\theta(-y + \frac{\partial \psi}{\partial x}) \\ \tau_{yz} &= G\theta(x + \frac{\partial \psi}{\partial y}) \end{aligned} \qquad (3)$$

The conditions over (x, y) come from the equilibrium equations.

The first set

$$\sigma_{ij,j} + X_i = 0 \tag{4}$$

produces only one (the first two are automatically satisfied):

$$\psi_{,xx} + \psi_{,yy} = \nabla^2\psi = 0 \tag{5}$$

The boundary equilibrium

$$\overset{\nu}{T}_i = \sigma_{ij}\,\nu_j = \bar{X}_i \tag{6}$$

applied to the extreme sections produces the relation

$$M_t = \theta G \iint \left[(x\,\psi_{,y} - y\psi_{,x}) + (x^2 + y^2)\right] da \tag{7}$$

and, in the lateral boundaries, where

$$\underline{\nu}^T = (\cos\alpha,\ \sin\alpha,\ 0)$$

$$\overset{\nu}{T}_3 = 0 \tag{8}$$

it is necessary that

$$\tau_{xz}\cos\alpha + \tau_{yz}\sin\alpha = 0$$

or, using (3)

$$\left(-y + \frac{\partial\psi}{\partial x}\right)\frac{dy}{ds} - \left(x + \frac{\partial\psi}{\partial y}\right)\frac{dx}{ds} = 0 \tag{9}$$

This lat can be read

$$\frac{\partial\psi}{\partial x}\cos\alpha + \frac{\partial\psi}{\partial y}\sin\alpha = y\cos\alpha - x\sin\alpha \tag{10}$$

The r.h.s. is the product

$$(x,y)\begin{Bmatrix}-\sin\alpha\\ \cos\alpha\end{Bmatrix} = \underline{r}^T.\underline{t} \tag{11}$$

which is nothing more than the position vector projection over the boundary tangent.

The l. h. s. is

$$(\psi_{,x};\psi_{,y})\begin{Bmatrix}\cos\alpha\\ \sin\alpha\end{Bmatrix} = \nabla\psi.\underline{\nu} = \frac{d\psi}{d\nu} \tag{12}$$

Then, condition (9) is transformed to

$$\frac{d\psi}{d\nu} = \underline{r}.\underline{t} \qquad (13)$$

The conditions over the warping function are now those of a classical NEUMANN problem

$$\nabla^2\psi = 0$$

$$\frac{d\psi}{d\nu} = r\cos(\underline{r},\underline{t}) \qquad (14)$$

There are also two more alternatives of presentation. As a DIRICHLET problem, using a harmonic conjugated function of ψ or, a POISSON problem through the use of PRANDTL function.

In the following examples the NEUMANN possibility has been chosen because the easy comprehension of its physical meaning.

POTENTIAL REPRESENTATION

Equation (14) is one of the classical problems in potential theory. Its properties has been fully studied, and specially well know is the Green reciprocity theorem which can be found elsewhere (ref.18,19).

Given two functions, ψ and ϕ, defined over a bounded region D with volume-element dv and piecewise smooth boundary ∂D, the Green formula is

$$\int_D (\psi\nabla^2\phi - \phi\nabla^2\psi)\,dv = \int_{\partial D} \left(\psi \frac{d\phi}{d\nu} - \phi \frac{d\psi}{d\nu}\right) ds \qquad (15)$$

where it is assumed C^1 continuity in D $+\partial$ D and C^2 in D (ref.19).

One important particularization is obtained if

$$-\nabla^2\phi = 4\pi\delta(\underline{p},\underline{q}) \qquad (16)$$

p, q are two points in the region and $\delta(\underline{p}, \underline{q})$ the Dirac function defined by the properties

$$\delta(\underline{p},\underline{q}) = 0 \quad \text{if } \underline{p} \neq \underline{q}$$

$$\int_D \xi(\underline{q})\ \delta(\underline{p},\underline{q})\,dv = \xi(\underline{p}) \qquad \forall\ \xi(\underline{q})\ C\ D \qquad (17)$$

But

$$-\nabla^2 \frac{1}{r(\underline{p},\underline{q})} = 4\pi\delta(\underline{p},\underline{q}) \qquad (18)$$

as it is easily demonstrated in 3-Dimension. Then using all the previous results (15) can be written

$$-4\pi\psi(\underline{p}) = \int_D \frac{\nabla^2\psi}{r}\, dv + \int_{\partial D} \psi \frac{d(\frac{1}{r})}{d\nu}\, ds - \int_{\partial D} \frac{1}{r} \frac{d\psi}{d\nu}\, ds \qquad (19)$$

Which is the classical superposition of a spatial, double and single layer distributions.

If

$$\nabla^2\psi = 0 \qquad (20)$$

$$\psi(\underline{p}) = \frac{1}{4\pi} \int_{\partial D} \frac{1}{r} \frac{d\psi}{d\nu}\, ds - \frac{1}{4\pi} \int_{\partial D} \psi \frac{d}{d\nu} (\frac{1}{r})\, ds \qquad (21)$$

In two dimensions the fundamental solution is

$$\phi(\underline{p},\underline{q}) = \frac{1}{2\pi} \ln \frac{1}{r(\underline{p},\underline{q})} \qquad (22)$$

Equation (21) can be applied also when $\underline{p}$ lies in the boundary. In this case however the singularity of the fundamental solution presents some trouble. The classical solution of eliminating a small sphere of center $\underline{p}$ (see Courant, Hilbert, vol. II, pp. 256, 257) produces

$$C\, \psi(\underline{P}) + \int_{\partial D} \psi \frac{d\Delta}{d\nu}\, ds = \int_{\partial D} \Delta \frac{d\psi}{d\nu}\, ds \qquad (23)$$

where, in 3-D

$$C = \begin{cases} 4\pi & \text{for } \underline{P} \text{ in } D \\ 2\pi & \text{for } \underline{P} \text{ in } D \\ 0 & \text{for } \underline{P} \text{ outside } D. \end{cases} \qquad \Delta = 1/r$$

or in 2-D

$$C = \begin{cases} 2\pi & \text{for } \underline{P} \text{ in } D \\ \pi & \text{for } \underline{P} \text{ in } D \\ 0 & \text{for } \underline{P} \text{ outside } D \end{cases} \qquad \Delta = \ln \frac{1}{r}$$

only when there is a continous tangent at $\underline{P}$. If there is a conical vertex, C is the vertex angle.

Nevertheless the calculation of C will be done using a useful idea credited to CRUSE.

Supposing ψ is constant, (23) is transformed into

$$C\ \psi(\underline{P}) = -\int_{\partial D} \psi \frac{d\Delta}{d\nu}\, ds \qquad (24)$$

B.I.E. METHOD

Equation (23) is the basis for the discretization of the boundary.

Following the F.E.M. isoparametric idea of IRONS, the field variables (ψ; ψ_n) are supposed to vary in a predetermined form between "nodes" in the boundary. This one is also assumed to behave in the same manner used for the field variables.

Then, doing the substitutions

$$\psi = \left[N_1(\xi); N_2(\xi); N_3(\xi); \ldots\right] \begin{Bmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \vdots \end{Bmatrix}^e$$

$$q = \frac{d\psi}{d\nu} = \left[N_1(\xi); N_2(\xi); \ldots\right] \begin{Bmatrix} q_1 \\ q_2 \\ \vdots \\ \vdots \end{Bmatrix}^e$$

$$ds = J\, d\xi \qquad (25)$$

where ξ are natural coordinates in the element and J the Jacobian to pass from the S system to the ξ one, equation (23) transforms to a system

$$C_i\psi_i + \sum_{j=1}^{N} \left[\int_{\partial D_j} N_1 \frac{d\Delta}{d\nu} J\, d\xi;\ \int_{\partial D_j} N_2 \frac{d\Delta}{d\nu} J d\xi; \ldots\right] \underline{\psi}^e$$

$$= \sum_{j=1}^{N} \left[\int_{\partial D_j} N_1\, J d\xi;\ \int_{\partial D_j} N_2\, \Delta J d\xi\ ; \ldots\right] \underline{q}^e \qquad (26)$$

where the integrations are extended to every elemental boundary subdomain ∂D_j, and

$$\partial D = \sum_{j=1}^{N} \partial D_j \qquad (27)$$

The calculation of C_i is very easy if formula (24) is discretized.

$$C_{i\,i} = -\sum_{j=1}^{N} \left[\int_{\partial D_j} N_1 \frac{d\Delta}{d\nu} J d\xi \; ; \ldots \right] \underline{\psi}^e$$

$$C_i \psi_i = -\sum_{j=1}^{N} \left[I_1^j \psi_1 + I_2^j \psi_2 + \ldots + I_i^j \psi_i + \ldots + I_N^n \psi_N \right]$$

and as

$$\psi_1 = \psi_2 = \ldots = \psi_i = \ldots = \text{constant}$$

$$C_i = -\sum_{j=1}^{N} (I_1^j + I_2^j + \ldots + I_N^j)$$

The value of the coefficient on the diagonal can be calculated through the knowledge of the off-diagonal ones.

Introducing the term $C_i \Psi_i$ in summation of the l.h.s. equation (26) can be written as a system

$$\underline{H}\,\underline{\psi} = \underline{G}\,\underline{q} \qquad (28)$$

where the terms of $\underline{H}$ and $\underline{G}$ are known calculated matrices and $\underline{\Psi}$, $\underline{q}$ are vectors defining the field in the boundary.

On a Neumann problem $\underline{q}$ is known and can be obtained making

$$\underline{\psi} = \underline{H}^{-1}\,\underline{G}\,\underline{q} \qquad (29)$$

On a Dirichlet problem the datum is $\underline{\Psi}$

$$\underline{q} = \underline{G}^{-1}\,\underline{H}\,\underline{\psi} \qquad (30)$$

Finally, in a mixed problem an interchange of columns permits the formal assimilation to one of the precited cases.

All this procedures must be thought as symbolical statements. The resolution is generally accomplished by means of a Gauss elimination method.

Moreover, the method used in this paper, proceeds directly to the automatic direct assemblage of a system

$$\underline{F} = \underline{K}\ \underline{X} \tag{31}$$

where in $\underline{F}$ are collected all the known data and in $\underline{X}$ every unknown of the problem.

PLANE PROBLEM. LINEAR INTERPOLATION

In the resolution of torsion problems the region D is plane and the boundary ∂D is a closed curve.

The discretization used in this paper is of a linear interpolation kind with some improvements over other standard elements. (see, as a comparison ref. 17).

Nodes are the extreme points of rectilinear elements in which a linear variation of ψ is assumed

$$\psi_\xi = [N_1,\ N_2] \begin{Bmatrix} \psi_1 \\ \psi_2 \end{Bmatrix}^e \quad ; \quad q_\xi = \frac{d\psi}{d\nu}\Big|_\xi = [N_1,\ N_2] \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}^e \tag{32}$$

and

$$N_1 = -\frac{1}{2}(\xi-1)$$
$$N_2 = \frac{1}{2}(\xi+1) \qquad -1 < \xi < 1$$

are the well known Irons shape functions.

The j element runs from node j to node j + 1 so

$$\int_{\partial D_j} \psi \frac{d}{d\nu}(\ln\frac{1}{r})\,ds = \int_{\partial D_j} [N_1 - N_2] \begin{Bmatrix} \psi_j \\ \psi_{j+1} \end{Bmatrix} \frac{d}{d\nu}(\ln\frac{1}{r})\frac{L_j}{2}\,d\xi \tag{33}$$

where L_j is the element length.

Similarly,

$$\int_{\partial D_j} \ln\frac{1}{r}\frac{d\psi}{d\nu}\,ds = \int_{\partial D_j} \ln\frac{1}{r}[N_1; N_2] \begin{Bmatrix} q_j \\ q_{j+1} \end{Bmatrix} \frac{L_j}{2}\,d\xi \tag{34}$$

Equation (23) is now

$$2C_i\psi_i + \sum_{j=1}^{N} L_j \int_{\partial D_j} N_1 \frac{d}{d\nu}(\ln \frac{1}{r})\, d\xi;\ \int_{\partial D_j} N_2 \frac{d}{d\nu}(\ln \frac{1}{r})\, d\xi \left| \begin{Bmatrix} \psi_j^j \\ \psi_j^j \\ \psi_{j+1}^j \end{Bmatrix} \right. =$$

$$= \sum_{j=1}^{N} L_j \left| \int_{\partial D_j} N_1 \ln \frac{1}{r}\, d\xi;\ \int_{\partial D_j} N_2 \ln \frac{1}{r}\, d\xi \right| \begin{Bmatrix} q_j^j \\ q_j^j \\ q_{j+1}^j \end{Bmatrix} \qquad (35)$$

To avoid problems, equation (35) must be now seen in the light of the general situation.

In every point of the contour there are three un knowns: the potential ψ , an the two fluxes before q_B , and after q_A. In a Dirichlet problem only one datum is supplied; the use of equation 35 is not enough to determine the problem.

In a Neumann problem both q_B, q_A are known and the use of eq. 35 solves the question.

In a mixed situation one always knows two of the three unknowns and again eq.35 is the only neccesary tool.

Previous approaches solved the indetermination of Dirichlet problem by assuming

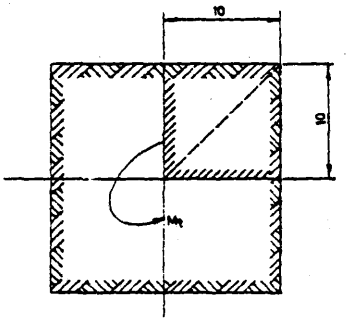
$$q_A = q_B$$

The inconsistence of this approach is only important in the case of sharp cornes and produces very bad results near them. Moreover, the q to be used in a sharp corner in Neumann case, is seldom known.

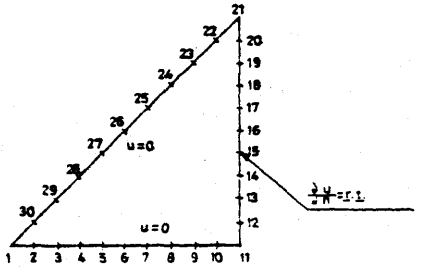
Brebbia and Dominguez (ref.17) circunwected this problem by using two nodes in every sharp corner, and then, truncating this one. The results are better than previously but, as would be expected, the Dirichlet condition is not well managed (see fig. 2 (d) ref.20).

The binode solution also increments artificially the number of equations to be solved.

Our approach has been then, the recognition of the diferent cases that can be found in a problem. In this light we think there are five cases to be considered round a node, i.e.,



PROBLEM DEFINITION



DISCRETIZACION

WARPING VALUES

Point	Theory	Present Method
11	0.0	0.0
12	3.477	3.49152
13	6.786	6.80869
14	9.756	9.78161
15	12.193	12.22441
16	13.894	13.92990
17	14.617	14.65840
18	14.075	14.11942
19	11.880	11.93549
20	7.555	7.55518
21	0.0	0.0

1) q_B ; q_A

2) q_B ; ψ

3) ψ ; q_A

4) sharp corner, ψ

5) smooth contour, ψ

and that, in every case, the key matter is the use of eq. 35.

In the cases 1, 2, 3 there are no problem because the remaining unknow will be obtained through the use of the final system of equations.

The problem of case 4 is that there are two unknowns q_B, q_A and only one equation to be used.

The question will be solved if each flux could be related to a common "dummy" variable. But it is well know that

$$q_B = \nabla\psi.\underline{\nu}_B = q \cos \alpha_B$$
$$q_A = \nabla\psi.\underline{\nu}_A = q \cos \alpha_A \qquad (36)$$

where q is the gradient modulus and $\underline{\nu}_B$, $\underline{\nu}_A$ the angles between the gradient and boundary normals, respectively before and after the node.

If a direction for the gradient is assumed, the only remaining variable is q and the situation is now the same as in cases 1, 2, 3.

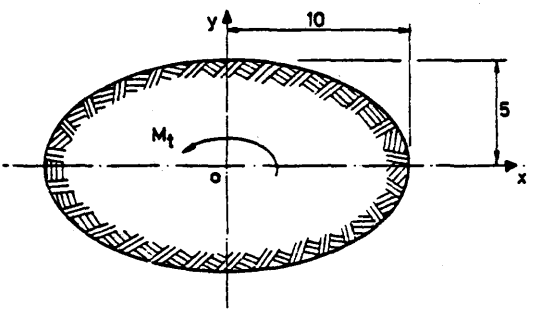
A guess, consistent with the linear hypotesis in the elements, is to imagine a plane passing through the neighbouring points in which the values of ψ are known.

If (i - 1) ; i, (i + 1) are three consecutive nodes and i is under study, it can be shown that the gradient of a plane defined by them is

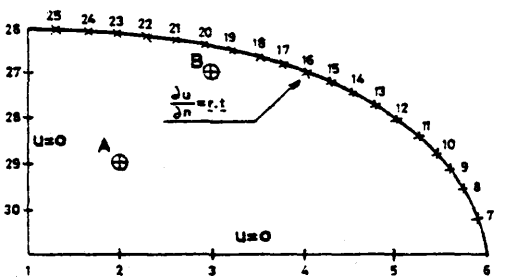
$$\left.\frac{\partial\psi}{\partial x}\right|_i = \frac{1}{\Delta_i}(y_i - y_{i+1}; y_{i+1} - y_{i-1}; y_{i-1} - y_i) \begin{Bmatrix} \psi_{i-1} \\ \psi_i \\ \psi_{i+1} \end{Bmatrix}$$

$$\left.\frac{\partial\psi}{\partial y}\right|_i = \frac{1}{\Delta_i}(x_{i+1} - x_i; x_{i-1} - x_{i+1}; x_i - x_{i-1}) \begin{Bmatrix} \psi_{i-1} \\ \psi_i \\ \psi_{i+1} \end{Bmatrix}$$

$$\Delta_i = (x_{i+1} - x_i)(y_{i-1} - y_i) - (x_{i-1} - x_i)(y_{i+1} - y_i) \qquad (37)$$



PROBLEM DEFINITION



⊕ INTERNAL POINTS
DISCRETIZATION

WARPING VALUES

Point	Theory	Present Method
7	-4.11495	-4.18318
9	-10.69080	-10.70310
11	-13.35163	-13. 33278
13	-14.80094	-14.79328
15	-14.88622	-14.88190
17	-13.91716	-13.91553
19	-11.99981	-11.99936
21	- 9 .35628	-9.35799
23	- 5 .95757	-5. 95942
25	-1. 96614	- 1.96842
A	-2.4	-2.400
B	-9.6	-9.598

NORMAL DERIVATIVES

1	0.00	0.00669
3	2.4	2.40176
5	4.8	4.86250
27	2.4	2.40254
29	1.2	1.20118
30	0.6	0.60076

where (x_j , y_j) are the coordinates of the nodes and φ_j the potential value in it.

The gradient direction is, then

$$\underline{n} = \frac{\nabla\psi}{|\nabla\psi|} = \left(\frac{\psi_{,x}}{\sqrt{\psi_{,x}^2 + \psi_{,y}^2}} \; ; \; \frac{\psi_{,y}}{\sqrt{\psi_{,x}^2 + \psi_{,y}^2}} \right) \qquad (38)$$

Using $\underline{n}$ one can compute α_B , α_A and, solving the N equations, determine the q value.

A final application of eq. (36) will show the desired values q_B , q_A.

Of course an iterative process could refine the guess but the run of several examples has shown that very good results can be obtained in this way.

Finally, case 5 is a particular situation in which the obvious solution is $q_A = q_B$. This same alternative is chosen in case number 4 when

$$\psi_{i-1} = \psi_i = \psi_{i+1}$$

because now the boundary is a level line of the potential surface.

DIRECT ASSEMBLING METHOD

Once established the system of equations (35) it is possible to write

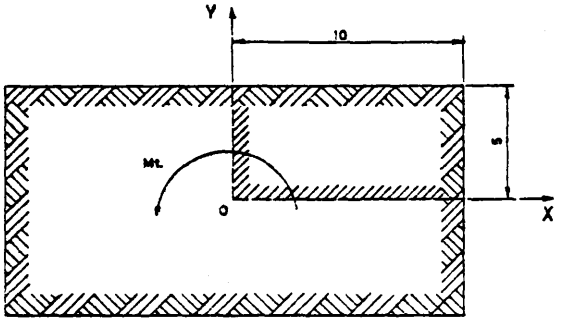
$$\underline{H}\,\underline{U} = \underline{G}\,\underline{Q} \qquad (39)$$

Reordering the equations in such a way that all the unknowns are on the l.h.s., one can write

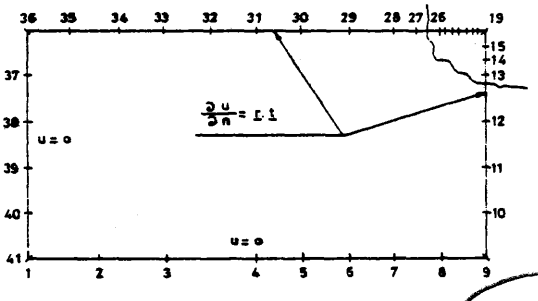
$$\underline{K}\,\underline{X} = \underline{F} \qquad (40)$$

where $\underline{X}$ is the vector of unknowns.

This process requires the storage of, two n x n matrices (when hypotesis $q_A = q_B$ in done) but one, really, only requires one as it is seen in eq.40. Our alternative is to form directly eq. (40) by recognizing the contribution of every element to each row of system (39). When element j is viewed from node i there are four contributions to matrices $\underline{H}$ and $\underline{G}$. The first two, produced by the two potentials φ_j , φ_{j+1} of the extreme nodes, are located in H. The



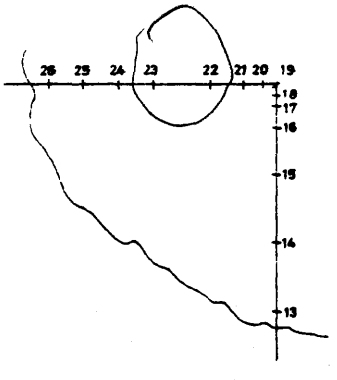
PROBLEM DEFINITION



DISCRETIZATION

Dirichlet

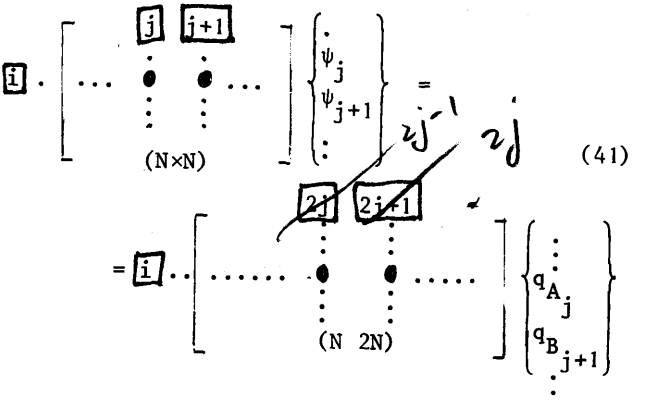
Fluxes



WARPING VALUES

Points	Theory	Pres. Method
9 aft.	0.0	0.00000
11	2.0	1.99997
13	4.0	3.99992
15	4.6	4.59991
17	4.9	4.89838
19 bef.	5.0	4.90387
19 aft.	-10.0	-9.90387
21	-9.85	-9.84777
23	-9.45	-9.45336
25	-9.15	-9.15399
27	-8.50	-8.50008
29	-7.00	-6.99996
31	-5.00	-4.99998
33	-3.00	-3.00001
35	-1.00	-1.00003
36 bef.	-0.00	-0.00000

another pair, contributed by q_A of node j and q_B of node j + 1, is in $\underline{G}$. The general scheme is then formation of a n x n matrix and a (n x x 2n) one,



At the price of a careful programming one can form the product of every coefficient by its corresponding data and put it in vector $\underline{F}$. If the code of the point under study indicates the presence of an unknown this is relocated in the adequate row of $\underline{X}$ and so is its correlated coefficient in matrix $\underline{K}$.

Repeating this process for every node one finds finally the system (40) which can be solved by standard methods.

EXAMPLE S

The method has been tested through various potential problems, and has shown a very good score.

Because of the paper title we only present the results obtained with torsion problems.

Fig. 1 shows the results of the torsion of a square prism. Using symmetrie one is able to reduce the problem to the study of an octant of the section, because the warping is zero in the coordinates and diagonal axis.

The results compare very well with the old values of SAINT--VENANT. In this respect it must be remermbered that the latter were obtained after the truncation of a series. Also it must be pointed out that our results have been obtained using single precision.

In the origin one has a point with the Dirichlet condition. The situation has bean very well approached, as it is shown by the results.

Fig. 2 shows the results obtained with an ellipse. In this occasion the warping function is nil only in the coordinate axis. Then

the fourth part of the domain has been discretized. The results are now compared with the analytical solution, and as can be seen, are good enough. Nevertheless the aproximation near the corner with vertical tangent is not as good in the middle indicating that a better discretization near this corner would be necessary.

This topic is very well shown in the third problem. Fig. 3 shows a rectangle which fourth was studied.

The direct conditions of analysis (i.e. $\psi = 0$ in the coordinates boundaries, $d\psi/d\nu = \underline{r} \cdot \underline{t}$ in the remaining ones) are very well analysed with only a saparse discretization.

The situation changes when, using these results, one tries to study a pure Dirichlet problem. Fig. 4 shows the discretization that must be used to obtain good results near the exterior corner.

CONCLUDING REMARKS

A general theory taht describes the B.I.E. linear approximation in potential problems is developed. The general equations are written and the different boundary conditions analysed. A method to treat the Dirichlet sharp vertex condition is presented. Though the study is developed for linear elements, its extension to higher order interpolation is straightforward.

A direct assembling procedure for the system of equations to be solved, is also described.

Finally some examples, rune in a Hewlwtt-Packard,21 MX computer, are shown.

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